Build up the radar equation…

If a radar transmits peak power $P_t$ from a point source isotropically (equally in all directions), the power density [W m$^{-2}$] is

$$D_p = \frac{P_t}{4\pi r^2}.$$ 

A small area $A_t$ intercepts the amount of power given by

$$P_\sigma = \frac{P_tA_t}{4\pi r^2}.$$ 

Radar antennas are NOT isotropic radiators! We introduce the concept of “gain,” which is the (nondimensional) ratio of the peak power at the beam maximum to that of an isotropic radiator.

$$P_\sigma = \frac{GP_tA_t}{4\pi r^2}.$$
Now, assume the target at range $r$ scatters the transmitted energy isotropically:

$$P_r = \frac{P \sigma A_e}{4\pi r^2},$$

where $A_e$ is the receiving area (aperture) of the antenna. Substituting,

$$P_r = \frac{G P_t A_t A_e}{(4\pi r^2)^2}$$

Gain and aperture are related by

$$G = \frac{4\pi A_e}{\lambda^2}.$$

Solve for $A_e$ and substitute…

$$P_r = P_t \frac{G^2 \lambda^2}{(4\pi)^3} \frac{1}{r} A_t$$
• $P_t$, $G$, and $\lambda$ are hardware characteristics for the radar. $r$ is range, and $A_t$ is the target.

• The target with area $A_t$ is not likely to scatter radiation isotropically.

• For many atmospheric scatterers, forward and backward scattering are favored — not isotropic!

• We introduce $\sigma$, the backscatter cross section.

The radar equation for a single target of backscatter cross section $\sigma$ is

$$P_r = P_t \frac{G^2 \lambda^2}{(4\pi)^3 r^4} \sigma$$

For some cases $\sigma$ can be determined theoretically or empirically.

Now consider distributed scattering targets (e.g. rain shaft).
We might imagine that the backscatter cross section for a volume is simply the sum of the backscatter cross sections of all the particles within the volume, i.e.

\[ P_r = P_t \frac{G^2 \lambda^2}{(4\pi)^3 r^4} \sum \sigma. \]

- However, this assumption isn’t strictly true for a single radar pulse!
- Scattering elements move relative to each other, and there are turbulent fluctuations across the resolution volume → instantaneous power return will not give an adequate measure of \( \Sigma \sigma \).
- Need more samples!
- Time between samples is critical

Too long: scatterers might be totally different
Too short: return might be virtually identical
• Want some degree of independence between measurements

• Decorrelation time — time it takes for a volume to decorrelate to a value of 0.01 from perfect correlation (i.e. time to go from 1.0 → 0.99)

• \( t_{0.01} \) [ms] ranges from \( 2\lambda \) to \( 3\lambda \) [cm]. \( t_{0.01} = 10 \) ms for 5 cm radar

• Short wavelengths “see” change in radar volume (redistribution of scatterers) more quickly → Decorrelation times are shorter for short wavelengths

• Wide array of particle sizes decreases the decorrelation time:
  
  Volume contains drizzle drops nearly all the same size → long \( t_{0.01} \).
  Volume contains mixed rain and hail → short \( t_{0.01} \).

• Strong turbulence tends to decrease the decorrelation time.

• For PRF of 1000 Hz, many pulses have to be averaged to get just a few independent samples.
So now we can be happy with $P_r = P_t \frac{G^2 \lambda^2}{(4\pi)^3 r^4} \sum \sigma$.

Let the backscatter volume be given approximately by

$$V = \pi \left( \frac{r \theta}{2} \right)^2 \frac{h}{2}, \quad h = c \tau, \quad \theta = \text{beam width}$$

Combine the two equations to get

$$\bar{P}_r = P_t \frac{G^2 \lambda^2}{(4\pi)^3 r^4} \pi \left( \frac{r \theta}{2} \right)^2 \frac{h}{2} \eta,$$

where $\eta \ [\text{m}^2 \ \text{m}^{-3}]$ is backscatter cross section per unit volume.
• This equation assumes that the signal transmitted power is the same over the entire beam, when in reality $P_t$ is Gaussian across the beam.

• An average value of gain should be used instead of the maximum (axial) value as it is defined.

Assuming a Gaussian beam pattern, the equation becomes

$$\bar{P_r} = P_t \frac{G^2 \lambda^2 \theta^2 h \eta}{1024 \pi^2 \ln 2 r^2}.$$  

• This differs from the non-Gaussian beam by a factor of $1/(2 \ln 2)=0.72$.

Backscatter cross section for a particle of radius $r_0 < 0.1\lambda$ is

$$\sigma = 64 \frac{\pi^5}{\lambda^4} |K|^2 r_0^6,$$

where $K$ is a function of the complex index of refraction of a sphere.
• Rayleigh scattering

Good for most raindrops and snowflakes but not for hailstones
More accurate at longer radar wavelengths

• $K$ depends on temperature, wavelength, and composition.

$|K|^2 \approx 0.93$ for liquid water

$|K|^2 \approx 0.21$ for ice

• Ice sphere has a radar cross section $2/9$ that of same-sized water sphere.

For a collection of spherical liquid drops under Rayleigh scattering,

$$\overline{P_r} = P_t \frac{G^2 \lambda^2}{(4\pi)^3 r^4} \frac{\pi^5}{\lambda^4} |K|^2 \sum r_0^6$$
In terms of drop diameters,

\[
\bar{P}_r = P_t \frac{G^2 \pi^5}{(4\pi)^3 r^4 \lambda^2} |K|^2 \sum D^6.
\]

Mean power received is determined by 1. radar parameters, 2. range, 3. value of \(|K|^2\), and 4. \(\Sigma D^6\).

Introduce a new quantity \(Z\), radar reflectivity, defined by

\[
Z = \sum V D^6 = \int_0^\infty N(D) D^6 dD
\]

Correcting for the Gaussian beam, the radar equation becomes

\[
\bar{P}_r = \frac{\pi^3 c}{1024 \ln 2} \left[ P_t \tau G^2 \theta^2 \frac{|K|^2}{r^2} \right],
\]

where the first term in brackets contains radar parameters, with target parameters in the second.
A rearrangement and further simplification…

\[ Z = \frac{2^{-r_P r C_r}}{|K|^2}. \]

\( P_r \) and \( Z \) vary over many orders of magnitude, so logs relative to a reference are convenient.

Return power: dBM (decibels relative to a mw)

Reflectivity: dBZ (decibels relative to 1 mm\(^6\) m\(^{-3}\))

One more thing: 10\( \log Z = 20 \log r + 10 \log P_r + C \) (slightly different from 11.9). Radar parameters only enter reflectivity through a simple offset.

Relating \( Z \) and precipitation rate

\[ Z = \int_{0}^{\infty} N_0 e^{-\lambda D} D^6 dD = N_0 \frac{\Gamma(7)}{\lambda^7}. \]

Remembering that \( \lambda = 41 R^{-0.21} \) for M-P, we get

\[ Z = N_0 \frac{\Gamma(7)}{(41)^7} R^{1.47}. \]
This expression between Z and R is in reasonable agreement with observational data that show

\[ Z = 200R^{1.6}. \]

Values of Z for several rain rates:

<table>
<thead>
<tr>
<th>R (mm h^{-1})</th>
<th>0.1</th>
<th>1</th>
<th>10</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z (mm^6 m^{-3})</td>
<td>5</td>
<td>200</td>
<td>7950</td>
<td>316000</td>
</tr>
<tr>
<td>dBZ</td>
<td>7</td>
<td>23</td>
<td>39</td>
<td>55</td>
</tr>
</tbody>
</table>

- This is one of literally hundreds (thousands?) of “Z-R” relationships used to retrieve a fundamental meteorological quantity (rainfall) from a not-so-fundamental quantity (reflectivity).

- Z-R relationships vary with precip. type (e.g. frontal, convective, stratiform), season, location, and anything else you can think of.

- Crude but useful.
Doppler

1. Radar sends out pulse of EM at frequency $f_0$, wavelength $\lambda$, and phase $\phi_0$
2. Encounters target at distance $r$ and returns back to antenna
3. Distance travelled is $d = 2r$, corresponding to $n = 2r/\lambda$ wavelengths and a phase shift of $\Delta \phi = 2\pi n$
4. The phase of the received pulse is then $\phi = \phi_0 + \Delta \phi = \phi_0 + \frac{4\pi r}{\lambda}$.
5. If the target is moving away from the radar, and the radial distance $r$ is increasing $\rightarrow \frac{d\phi}{dt} = \frac{d\phi}{dr} \frac{dr}{dt} = \frac{4\pi dr}{\lambda dt}$.

- Particle may be moving at an angle to the beam direction, but the radial velocity $v = \frac{dr}{dt}$ is responsible for the beam phase shift.
Rate of change of the phase of a wave is just the angular velocity,

\[
\frac{d\phi}{dt} = \omega_d = 2\pi f_d.
\]

We equate to find the Doppler shift frequency,

\[
2\pi f_d = 4\pi v / \lambda, \text{ so } f_d = 2v / \lambda.
\]

Now, \(f_0 = 3 \times 10^9 \text{ s}^{-1} \left(\frac{c}{\lambda}\right)\), and \(\lambda = 10 \text{ cm}\) for the WSR-88D…

<table>
<thead>
<tr>
<th>(v (\text{ms}^{-1}))</th>
<th>(f_d (\text{s}^{-1}))</th>
<th>% of (f_0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>2</td>
<td>(2/3 \times 10^{-7})</td>
</tr>
<tr>
<td>1.0</td>
<td>20</td>
<td>(2/3 \times 10^{-6})</td>
</tr>
<tr>
<td>10</td>
<td>200</td>
<td>(2/3 \times 10^{-5})</td>
</tr>
<tr>
<td>100</td>
<td>2000</td>
<td>(2/3 \times 10^{-4})</td>
</tr>
</tbody>
</table>

Very small change as percentage of \(f_0\)! Measuring \(f_d\) requires stability and the ability to distinguish small differences in frequency.
Nyquist criterion — To digitally represent a waveform, the sampling rate has to be twice the frequency of the highest frequency in the input signal.

The sampling rate for a pulsed radar is the PRF, so

\[ f_{d_{max}} = \frac{PRF}{2} = \frac{2v_{max}}{\lambda}, \]

and the unambiguous velocity range is then

\[ |v_{max}| \leq \frac{PRF \cdot \lambda}{4}. \]

Higher velocities are aliased into the unambiguous range.

Another way of thinking about this is that for every pulse the radar measures the phase angle difference \( \Delta \phi_p \), which is between \(-\pi\) and \(\pi\). It is possible that the true phase angle is the measured phase angle plus an integral multiple of \(\pm 2\pi\). Velocity is unambiguous only if the true phase change lies between \(-\pi\) and \(\pi\), i.e. only if target moves less than a quarter of a wavelength between pulses.
Recall $v_{max} = \frac{PRF \cdot \lambda}{4}$ and $r_{max} = \frac{c}{2 \cdot PRF}$. Eliminate PRF and get the relation

$$r_{max} v_{max} = \frac{c \lambda}{8}.$$  

For $\lambda = 10$ cm, we may desire an unambiguous range of 200 km  

$\rightarrow v_{max} = 18.75$ m s$^{-1}$.  

If we wish to have $v_{max} = 50$ m s$^{-1} \rightarrow r_{max} = 75$ km.

- You can have a large unambiguous range or a large unambiguous radial velocity, but not both!
- Going to a longer wavelength will increase $r_{max}$ and $v_{max}$.
Velocity folding example #1:
Velocity folding example #2:
As you might imagine, a sampled volume will contain a distribution of particle shapes, sizes, and motions, so the phase difference will not be the same from pulse to pulse.

The radar will send out many pulses to sample the same volume and will build up a distribution of return power as a function of radial velocity.

The total returned power is obtained by integrating the velocity spectrum:

$$P_t = \int_{-\infty}^{\infty} S(v_r)dv_r.$$  

The mean radial velocity is just the power-weighted first moment of the velocity spectrum:

$$\bar{V}_r = \frac{\int_{-\infty}^{\infty} v(r)S(v_r)dv_r}{\int_{-\infty}^{\infty} S(v_r)dv_r}.$$
The breadth of this spectrum can be an indication of
1. wide range of particle types
2. strong turbulence
3. strong shear

The second moment of the Doppler spectrum gives a variance, which is called spectral width:

\[ \sigma^2 = \frac{\int_{-\infty}^{\infty} v^2(r) S(v_r) dv_r}{\int_{-\infty}^{\infty} S(v_r) dv_r} \]

**Multiparameter radar (dual or circular polarization, perhaps multiple \( \lambda \))**

Polarization of an EM wave is defined by the direction of the electric field vector. In a dual polarization radar, the transmitted pulses alternate between vertical and horizontal polarization.
Why?
Because the backscatter cross sections for different precip. types can depend on polarization.

One useful parameter obtained from linearly polarized radars is

$$Z_{DR} = 10 \log \left( \frac{Z_H}{Z_V} \right),$$

which can be used as an indication of the aspect ratio of the particles.

For small droplets, $Z_{DR}$ tends to be near 0. As they grow and become more oblate, $Z_H > Z_V$, and $Z_{DR}$ increases ($Z_{DR} \sim 4.5$ for a 4 mm drop).

Small values of $Z_{DR}$ mean either spherical drops or a random orientation of a population of non-spherical particles.

Consider $Z_{DR}$ of: Rain? Snow? Graupel? Hail?
Another important dual polarization parameter

Recall that the phase velocity for EM waves is

\[ v_p^2 = \frac{1}{\mu \varepsilon}, \]

and if \( \varepsilon = \varepsilon_0 \) and \( \mu = \mu_0 \), then \( v_p = c \).

However, for liquid water \( \varepsilon = 80\varepsilon_0 \), and

\[ v_p^2 = \frac{1}{\mu_0(80\varepsilon_0)} = \frac{1}{80\mu_0\varepsilon_0} = \frac{c^2}{80}, \]

so \( v_p = 3.35 \times 10^7 \text{ m s}^{-1} \).

- The more liquid water the signal travels through, the more it’s slowed.

- If a sample volume has many large rain drops, a horizontally polarized pulse “sees” more liquid water than a vertically polarized pulse.

This creates a phase lag, \( \phi_{DP} = \phi_V - \phi_H \), and a parameter known as KDP, which is the rate of change of this phase lag with range,

\[ K_{DP} = \frac{d}{dr}\phi_{DP}. \]

As for ZDR, KDP is quite small for spheres or random-orientations.