One explanation for the negative exponential (M-P) distribution of raindrops is drop breakup.

Drop size is limited because — increased chance of disruption with size

Drop breakup:

- Spontaneous
- Collisional
Spontaneous Breakup:

- Arises from aerodynamically-induced circulation in the drop
- Komabayasi et al. (1964) show:
  1. This mode of breakup is a function of drop size and spectrum of small droplets ultimately produced
  2. Surface tension may not be able to maintain a 3 mm (diameter) drop
  3. 6 mm drops (diameter) are unstable and exist only briefly

- Theoretical calculations give flatter drop spectra than those observed, implying an *overestimate* of breakup

- More recent studies show that drops may become as large as 10 mm before this mode of breakup occurs → maybe spontaneous breakup is *not* important in atmospheric situations
Collisional Breakup

- Lab study: water drops between 0.3 and 1.5 mm colliding at an angle with relative velocities between 0.3 and 3 m s\(^{-1}\)
- Ultimate coalescence is less likely for larger values of drop size, relative velocity, and impact parameter (??)
- Disruption when RKE of the drop > than surface energy required for separate drops
- Comparing these two energies gives a coalescence efficiency:

\[ \varepsilon = \frac{12\sigma f(R/r)}{5r\rho U^2}, \]

where \(\sigma\) = surface tension, \(\rho\) = density, \(U\) = relative velocity.
Drop breakup

\[ \varepsilon = \frac{12 \sigma f(R/r)}{5r \rho U^2} \]

\[ \varepsilon < 0.2 \] exist for \( 1 \text{ mm} < R < 2.5 \text{ mm} \)

**Curiosity:** Note \( \varepsilon \) is not symmetric across \( R=r \)

**FIG. 10.3.** Coalescence efficiency from the theory of Brazier-Smith *et al.*, as a function of drop radii. The value of \( \varepsilon \) is the fraction of collisions that result in permanent coalescence. All collisions lead to coalescence outside the contour \( \varepsilon = 1 \).
To really figure this out, would have to go back to the source articles:


(Good luck finding the first one, should you ever need it!)
Collision efficiency in 10.3 (left) is unity where coalescence efficiency in 8.3 (right) is less than unity, and conversely

\[ \therefore \text{Collection efficiency approx. equals min}[\text{collision } \varepsilon, \text{ coalescence } \varepsilon] \]
Rogers and Yau Chapter 10 — Drop breakup

- B-S et al. showed that the collisions that didn’t coalesce produced from 1-10 small satellite drops from 20-200 µm radius
- Approximated this effect by assuming that three satellite drops would be produced, each having volume $0.04V_1V_2/(V_1 + V_2)$

Young (1975) put it all together — growth by condensation and coalescence, and collision-induced drop breakup.

Calculated DSD in a cloudy volume rising with constant $w$, modeling drop breakup as B-S et al.
30 minutes of ascent

\[ w = 3 \text{ m s}^{-1} \]

maritime cloud

Marshall-Palmer!
• A later study (Low and List 1982) argued that the fall velocities and forces (gravity, drag) should all be in the vertical plane so the drops have the proper relative velocities between them and the air.

• Carried out experiments and noted three distinct breakup types
Collision tore off one side of large drop. Coalescence occurs temporarily but disk spreads and breaks up. Drop identities mostly preserved most like B-S. 

FIG. 10.5. Schematic illustration of the three common types of drop breakup. (From McTaggart-Cowan and List, 1975.)
Breakup occurrence and type depends on point of impact but also collision kinetic energy (CKE) of the colliding droplets:

\[ CKE = \left( \frac{2}{3} \pi \rho L \right) \frac{R^3}{1 + \gamma^3} U^2, \]

where \( U \) is relative velocity and \( \gamma = \frac{R}{r} \).

- All three modes exist at large CKE, depending upon impact point
- At small CKE and small \( \gamma = \frac{R}{r} \rightarrow \) neck/filament breakup
- At small CKE, \( E_T = CKE + \) surface energy excess from 2drops \( \rightarrow 1 \). Coalescence will occur if \( E_T \) is dissipated through oscillation and deformation of drop
Low and List (1982) obtained the empirical relation for coalescence $\varepsilon$,

$$
\varepsilon = a \left( \frac{\gamma}{1 + \gamma} \right)^2 \exp \left[ -\frac{b \sigma E_T^2}{S_c} \right] \text{ for } E_T < 5 \mu J
$$

and 0 otherwise.

As before,

$$
E_T = CKE + 4\pi \sigma (r^2 + R^2) - r\pi \sigma (r^3 + R^3)^{2/3}
$$

- surface energy of separate drops
- surface energy of coalesced drop
Results from Low and List (1982):

Minimum $\varepsilon$ goes roughly along $R = 2r$, near max CKE region. Unlike B-S, here large $r$ is associated with small $\varepsilon$.
Rogers and Yau Chapter 10 — Drop breakup
Collision of drops — initial diameters of 0.46 and 0.18 cm (BIG drops)

Neck/Filament breakup

Note that most of the character of the initial drops remains.
Sheet breakup

- New small drops
- 18 µm mode eliminated
- Little change in large particle mode
Rogers and Yau Chapter 10 — Drop breakup

Disk breakup

New large population of small drops

Eliminates or at least reduces growth of large drops
Studies in the mid-80s calculated DSD as it evolves from initial M-P:

1. Reaches equilibrium DSD eventually
2. Shape of DSD stabilizes
3. Large drops grow by accreting small ones
4. Large drops break up, replenishing small droplet population

Eq. spectra definitely NOT M-P!
Trimodal
Unlikely to see this trimodal dist in atmos.
“Snow” crystals are highly variable (aggregates of crystals or small dendrites; large vapor-grown dendrites) so use mass or melted diameter for distribution

Typical size distribution:

\[ N_D = N_0 e^{-\Delta D} \]

\[ N_0 \text{ (m}^{-3} \text{ mm}^{-1}) = 3.8 \times 10^3 R^{-0.87} \]

\[ \Delta \text{ (cm}^{-1}) = 25.5 R^{-0.48} \]

\[ D_0 \text{ (cm)} = 0.144 R^{0.48} \]

- Larger value of \( R \)
- Smaller value of \( R \)
Precip. rate is flux of water volume through a horizontal surface:

\[ R(D) = N(D) \frac{\pi}{6} D^3 u(D) \quad [m^3 m^{-2} s^{-1} m^{-1}] \]

To get total, integrate over all drop sizes:

\[ R = \frac{\pi}{6} \int_0^\infty N(D) D^3 u(D) dD \]

Units are \([m^3 m^{-2} s^{-1}]\) (volume of water flowing through a surface) but more commonly expressed as \([\text{mm h}^{-1}]\).

Similarly, liquid water content \([\text{kg m}^{-3}]\) is calculated by

\[ L = \frac{\pi}{6} \rho L \int_0^\infty N(D) D^3 dD \]

Note that LWC is independent of fall speed.
Bulk microphysical models

We’ve discussed in this class the fundamental processes for growth of liquid and solid microphysical particles. Let’s see how these ideas might be incorporated into numerical models.

Why *bulk* schemes?

• The models discussed in R&Y Ch 8 predict evolution of drop spectra
• Splits drop spectrum into bins and solves water continuity equations
• ~25 bins is a typical number
• VERY computationally expensive: k equations (e.g. 25), ~k^2 operations to calculate coalescence
• Imagine doing this over a 3D NWP or cloud model grid! Ouch!
Bulk microphysical models

What is a bulk scheme (model, parameterization)?

• Bulk models assume only a few categories of water substance
• Minimizes number of conservation equations
• Minimizes number of calculations
• Disadvantages: lots of assumptions involved, which may be violated. Using a bulk model may indeed be a “risky scheme!”
Bulk microphysical models

Simple bulk model of Kessler (1969)

- Liquid water only (i.e. warm rain)
- Two categories:
  - cloud water — small droplets; “monodisperse”; zero terminal velocity
  - rain water — M-P distribution

Bulk models will in general have conservation equations like

$$\frac{dq_i}{dt} = SOURCES_i - SINKS_i$$
Bulk microphysical models

For the Kessler scheme, the conservation equations are

\[ \frac{d l_c}{dt} = S - A - C + D_{l_c} \]

\[ \frac{d l_R}{dt} = \frac{1}{\rho} \frac{\partial}{\partial z} (\rho V l_R) + A + C - E + D_{l_R} \]

\[ \frac{dq_v}{dt} = -S + E + D_R \]

S — source of cloud water due to condensation
A — “Autoconversion” stochastic coalescence process converting cloudwater → rainwater
C — accretion of cloudwater by rainwater
D — turbulent diffusion of various quantities
V — mass-weighted terminal velocity (defined positive)
E — evaporation of rainwater

Consider A, C, and the precip. flux terms.
**Bulk microphysical models**

**A — Autoconversion**

- The random and very ill-understood process by which small cloud droplets produce a few larger, precipitating droplets is called autoconversion.

- Autoconversion formulations typically are quite *ad hoc* and may be subject to much criticism.

For the Kessler scheme,

\[
A = H(l_c - l_{c0})K(l_c - l_{c0})
\]

K has units of [s\(^{-1}\)].

\[A = 0 \text{ kg kg}^{-1} \text{ s}^{-1}\] for values below the \(l_{c0}\) threshold

\[l_{c0} = 1 \text{ kg kg}^{-1}\] in original paper, corresponding to the threshold value of cloudwater where Kessler began seeing precipitation formation
Bulk microphysical models

Collection — C

• Collection of cloud droplets by falling rain drops

Accretion rate equation (may be obtained from R&Y 8.15 or 9.8)

\[ \frac{dM_D}{dt} = \rho \pi \frac{D^2}{4} \epsilon_D V_D l_c \]

RHS is a function of size.

We assume M-P distribution

\[ N = N_0 e^{-\lambda D} \]

Integrate over the M-P DSD
Bulk microphysical models

Integrate over the M-P DSD:

\[
\frac{dl_R}{dt} = \int_0^\infty \frac{\pi}{4} D^2 \varepsilon_D V_D l_c N_0 e^{-\lambda D} dD
\]

Need \( V_D \) as a function of \( D \). Use drag on sphere for high Reynolds #s:

\[
V_D = k \left( g \frac{\rho_l}{\rho} \right)^{1/2} D^{1/2}
\]

Substituting,

\[
C = \frac{dl_R}{dt} = kg^{1/2} \left( \frac{\rho_l}{\rho} \right)^{1/2} \frac{\pi}{4} \varepsilon N_0 l_c \int_0^\infty D^{5/2} e^{-\lambda D} dD
\]
Bulk microphysical models

Because we used M-P, the integral has an analytic solution...

\[ C = k \left( \frac{\rho l}{\rho} \right)^{1/2} g^{1/2} \pi^{-1/4} \varepsilon N_0 l_c \frac{\Gamma(7/2)}{\lambda^{7/2}} \]

Want everything in terms of \( l_c \) and \( l_R \), not M-P intercept parameter!

Realize that total rainwater content \( l_R \) is integral of the mass of each drop over the DSD:

\[ l_R = \frac{\rho l \pi}{\rho} \frac{1}{6} N_0 \int_0^\infty D^3 e^{-\lambda D} dD = \frac{\pi \rho l}{6 \rho} N_0 \frac{\Gamma(4)}{\lambda^4} \]

Use this relation to find \( \lambda^{7/2} \) and plug into above expression for \( C \) to get

\[ C = k_1 g^{1/2} \left( \frac{\rho}{\rho_l} \right)^{3/8} \varepsilon N_0^{1/8} l_c l_R^{7/8} \]
Bulk microphysical models

WHEW!

\[ C = k_1 g^{1/2} \left( \frac{\rho}{\rho_l} \right)^{3/8} \varepsilon N_0^{1/8} l_c l_R^{7/8} \]

The point here is that the collection varies as the product of the mixing ratios, i.e.

\[ C \sim l_c l_R \]
Bulk microphysical models

An expression for precipitation flux is derived in a similar manner, taking $V_l R$, assuming a relation for terminal fall speed as a function of drop size, and integrating over the M-P distribution.

I’ll mercifully not show this one!

These bulk models get MUCH more complicated as species are added…
Bulk microphysical models (Lin et al. 1983)

FIG. 1. Cloud physics processes simulated in the model with the snow field included. See Table 1 for an explanation of the symbols.